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Nonlinear reduced fluid equations for toroidal plasmas

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Nonlinear reduced fluid equations are derived for studying resistive instabilities in large-aspect-ratio, low-beta toroidal plasmas. An ordering is developed in which plasma compressibility as well as the poloidal curvature are retained. The nonlinear equations can be linearized and used to reproduce the Mercier criterion in the large-aspect-ratio, low-beta limit. A second set of reduced equations is derived from the Braginskii fluid equations. These equations, which are very similar to the reduced magnetohydrodynamic equations, contain diamagnetic effects as well as parallel transport associated with magnetic fluctuations. Both sets of equations conserve energy exactly.

I. INTRODUCTION

Resistive tearing and ballooning modes are believed to play a dominant role in both the stability and energy confinement in tokamak discharges.^{1,2} During the past few years extensive numerical calculations have been carried out to develop an understanding of the nonlinear behavior of these instabilities.²⁻⁴ Most of this computational work has been based on the reduced magnetohydrodynamic (MHD) equations.^{5,6}

The reduced MHD equations were derived for the study of global ideal kink modes in a pressureless plasma⁵ and subsequently extended to include finite pressure for the study of pressure-driven modes.⁶ The advantages of using the reduced equations, rather than the full MHD equations, are twofold. First, the reduced equations are simpler, involving a few scalar quantities. Second, the reduced equations do not describe the compressional Alfvén wave so that the maximum step which can be taken is not limited to the time it takes an Alfvén wave to propagate one grid space across the magnetic field.⁷

The ideal reduced equations have been modified to study resistive modes by adding a term ηJ_{\parallel} to Ohm's law, where η is the resistivity and J_{\parallel} is the parallel current density. The resulting equations have been used to study the evolution of resistive pressure-driven modes.² However, it was subsequently realized that the linear theory of resistive modes based on the reduced equations did not correspond to that based on the full set of resistive equations.⁸ In particular, the effects of plasma compression, average favorable curvature, and parallel inertia, which can have a strong stabilizing influence on resistive modes,^{9,10} are not present in the reduced equations.

The original ideal reduced MHD equations were derived to study global modes with large growth rates. Specifically, it was assumed that the perpendicular scale length of perturbations was of the order of a , the minor radius of the tokamak, and that the time scale for the evolution of perturbed quantities was of the order of the Alfvén time, $\tau_A \equiv R/c_A$, where R is the major radius and c_A is the Alfvén speed. While these assumptions are valid for global kink and ballooning modes they are not valid for resistive modes which evolve more slowly and are usually strongly localized in minor radius. Resistive modes are inherently weaker instabilities, are much more sensitive to the detailed magnetic

geometry, and therefore require more physical effects to properly describe. The *ad hoc* inclusion of resistivity in the ideal reduced equations does not lead to a consistent set of reduced equations for resistive modes. In this manuscript we expand the fluid equations with an ordering which is appropriate for resistive modes, namely,

$$\frac{\partial}{\partial t} \ll \tau_A^{-1},$$
$$\nabla_{\perp} \gg a^{-1},$$

where \perp refers to the direction perpendicular to the magnetic field \mathbf{B} . As in Strauss' original derivation of the reduced MHD equations, the fast Alfvén modes are eliminated by neglecting inertia in the component of the one-fluid momentum equation which describes the plasma compression. One is then left with an equation for local force balance (total pressure, magnetic and thermal, balancing the magnetic stress). As a consequence, the plasma is incompressible to lowest order although we show that corrections due to finite compressibility must be retained when investigating resistive modes. The elimination of these fast modes is fundamental to the development of a set of reduced equations.

Two sets of reduced equations are derived: the first from the resistive MHD equations and the second from the Braginskii fluid equations.¹¹ Both sets of equations conserve energy exactly and can also be used to reproduce the Mercier criteria for a large-aspect-ratio torus, i.e., average favorable curvature for $q = B_{\phi} r / R B_{\theta} > 1$.¹²

The reduced resistive MHD equations, which are derived in Sec. II, have five independent variables: the stream functions Φ and ψ , the pressure p , the mass density ρ , and the parallel flow v_{\parallel} . For a constant mass density $\rho = Mn$, where M is the ion mass, n drops out so v_{\parallel} is the only new variable not in Strauss' original equations. Finite compressibility as well as the curvature of the poloidal field are retained.

The reduced Braginskii equations, which we derive in Sec. III, are very similar to the resistive MHD equations of Sec. II. The Ohm's law, which comes from electron force balance, now contains pressure terms which in linear theory cause the diamagnetic rotation of drift waves and drift-tearing modes.¹³ An equation for the electron temperature $T_e(T_i = 0)$ replaces the pressure equation which appeared in the resistive equations of Sec. II. Parallel thermal conduction is included in this equation for T_e so that transport re-

sulting from magnetic fluctuations is self-consistently included and does not have to be added in an *ad hoc* manner. It has recently been shown that ∇T_e has a stabilizing influence on the linear tearing mode.¹⁴ The linearized versions of our nonlinear equations in toroidal geometry can be used to reproduce this stabilizing effect.

II. RESISTIVE MHD EQUATIONS

We begin with the equations of compressible, resistive MHD,

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{(\mathbf{J} \times \mathbf{B})}{c}, \quad (1)$$

$$\mathbf{E} + (\mathbf{v} \times \mathbf{B})/c = \eta \mathbf{J}, \quad (2)$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{J}/c, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (5)$$

$$\frac{dp}{dt} + \gamma_s p \nabla \cdot \mathbf{v} = (\gamma_s - 1) \eta \mathbf{J}^2, \quad (6)$$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad (7)$$

where ρ is the mass density, p is the pressure, \mathbf{E} and \mathbf{B} are the electric and magnetic fields, \mathbf{v} is the fluid velocity, \mathbf{J} is the current density, γ_s is the ratio of specific heats, and η is the resistivity. The total time derivative is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \quad (8)$$

As discussed in the Introduction, we are interested in following time scales which are much longer than compressional Alfvén time,

$$\frac{\partial}{\partial t} \ll c_A \nabla_{\perp}.$$

In this limit the largest terms in the momentum balance equation in the directions perpendicular to the magnetic field describe static force balance, that is

$$0 \simeq \nabla_{\perp} (p + B^2/8\pi) - (1/4\pi) B^2 \kappa, \quad (9)$$

where we have used Eq. (3) to eliminate \mathbf{J} and have defined the field line curvature,

$$\kappa = \mathbf{b} \cdot \nabla \mathbf{b}, \quad (10)$$

where $\mathbf{b} = \mathbf{B}/B$ is the unit vector in the direction of the magnetic field. By neglecting inertia in Eq. (9) we have eliminated the compressional magnetosonic wave. Taking the curl of Eq. (1) and forming the scalar product with \mathbf{b} , we obtain information about the noncompressible motion of the plasma

$$\frac{c}{B} \mathbf{b} \cdot \nabla \times \rho \frac{d\mathbf{v}}{dt} = \mathbf{B} \cdot \nabla \frac{J_{\parallel}}{B} - \frac{c}{4\pi} \frac{\mathbf{b} \times \kappa}{B} \cdot \nabla B^2, \quad (11)$$

where $J_{\parallel} = \mathbf{b} \cdot \mathbf{J}$. The pressure balance relation in (9) then enables us to rewrite (11) as

$$c \frac{\mathbf{b} \cdot \nabla \times \rho}{B} \frac{d\mathbf{v}}{dt} = \mathbf{B} \cdot \nabla \left(\frac{J_{\parallel}}{B} \right) + 2c \frac{\mathbf{b} \times \kappa}{B} \cdot \nabla p. \quad (12)$$

Finally, the component of Eq. (1) in the \mathbf{b} direction gives

$$\rho \mathbf{b} \cdot \frac{d\mathbf{v}}{dt} = -\mathbf{b} \cdot \nabla p. \quad (13)$$

Thus, the momentum balance equation reduces to two equations that evolve the velocity, Eqs. (12) and (13), and one equation describing force balance, Eq. (9). Equation (12) basically describes the rotation of the fluid around \mathbf{b} , while Eq. (13) describes the motion along \mathbf{b} . Our description of \mathbf{v} is not yet complete since we have not yet determined $\nabla \cdot \mathbf{v}$. This information is obtained by requiring that the pressure and magnetic field evolve so as to maintain force balance, Eq. (9).

We now examine Ohm's law, Eq. (2). In the low-frequency limit ($\partial/\partial t \ll c_A \nabla_{\perp}$), the plasma motion is nearly incompressible and is driven dominantly by the $\mathbf{E} \times \mathbf{B}$ drift produced by the electrostatic potential Φ (this is shown explicitly later). To lowest order, Eq. (2) reduces to

$$\mathbf{v}_{\perp} = -c \nabla \Phi \times \mathbf{b} / B. \quad (14)$$

As in our previous manipulations of the momentum equation, the dominant ($\nabla \Phi$) component of Eq. (2) is annihilated by taking the curl of the equation and dotting with \mathbf{B} ,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{v}_{\perp} \cdot \nabla \right) \frac{B^2}{2} - B^2 (\nabla \cdot \mathbf{v}_{\perp} + \kappa \cdot \mathbf{v}_{\perp}) \\ = -4\pi \eta \mathbf{J}^2 - c^2 \nabla \cdot \eta \nabla p, \end{aligned} \quad (15)$$

where we have used (5) to eliminate $\nabla \times \mathbf{E}$ and have approximated $\mathbf{J} \times \mathbf{B}/c \simeq \nabla p$ in the resistive term. Equations (6) and (15) formally evolve p and B^2 , respectively. However, since p and B^2 must also satisfy local force balance [Eq. (9)], Eq. (15) effectively determines $\nabla \cdot \mathbf{v}_{\perp}$. Thus, all three components of \mathbf{v} can now be computed. Our treatment of Ohm's law is completed by dotting with \mathbf{b} ,

$$E_{\parallel} = \eta J_{\parallel}. \quad (16)$$

In order to proceed further we introduce a specific ordering appropriate to the study of low toroidal mode number modes in a large-aspect-ratio tokamak. A difficulty in developing an ordering for resistive modes is reconciling the fundamental differences in the structure of perturbations in the region near rational surfaces where $\mathbf{b} \cdot \nabla$ is small and ∇_{\perp} is large and away from the rational surfaces where the perturbations are more global. We have found that it is necessary to retain more information to correctly model the plasma near the rational surfaces and we have adopted an ordering which is appropriate for this region. The resulting equations, of course, also correctly describe regions away from the rational surface.

We take as our fundamental expansion parameter ϵ , where

$$\epsilon = a/R \ll 1,$$

and R and a are the major and minor radii of the torus. We consider a low-pressure plasma with $\beta = 8\pi p/B^2$ given by

$$\beta \sim \epsilon^2. \quad (17a)$$

The magnetic field components in the toroidal direction (B_{ϕ}) and the poloidal plane (B_p) satisfy

$$B_p/B_{\phi} \sim \epsilon. \quad (17b)$$

The time derivative is assumed to scale as

$$\frac{\partial}{\partial t} \sim \frac{\epsilon^2}{\tau_A}, \quad (17c)$$

and the perpendicular and parallel gradients are assumed to fall in the ranges

$$(\epsilon^2 a)^{-1} \gtrsim \nabla_{\perp} \gtrsim a^{-1}, \quad (17d)$$

and

$$R^{-1} \gtrsim \nabla_{\parallel} \gtrsim \epsilon^2 R^{-1}. \quad (17e)$$

The fluid velocities v_{\perp} and v_{\parallel} are taken to scale as

$$\epsilon^3 c_A \gtrsim v_{\perp} \gtrsim \epsilon^5 c_A, \quad v_{\parallel} \sim \epsilon^3 c_A, \quad (17f)$$

and the resistivity is given by

$$\tau_A / \tau_r \sim \epsilon^6, \quad (17g)$$

where $\tau_r = 4\pi a^2 / \eta c^2$ is the macroscopic resistive diffusion time.

The motivation for this ordering is not obvious at first. The choice $\beta \sim \epsilon^2$ is appropriate for resistive modes since for $\beta \sim \epsilon$ ideal ballooning modes may be unstable,¹⁵ which is the case considered by Strauss.⁶ The condition $B_p / B_{\phi} \sim \epsilon$ is chosen so that the safety factor $q = r B_{\phi} / R B_p$ is of order unity, which is appropriate for tokamaks. The orderings for the time derivative, the spatial gradients, and the resistivity are basically identical to that which would be obtained in carrying out a linear stability analysis of Eqs. (1)–(7) and are somewhat more complex than in Strauss' reduced equations. The perpendicular gradient ∇_{\perp} actually has two components. In the direction perpendicular to the initial flux surfaces (along $\nabla \psi$, where ψ is the flux function), the spatial scale Δ of the perturbations is of order $\Delta \sim \epsilon^2 a$, while in the $\mathbf{b} \times \nabla \psi$ direction (essentially poloidal) $\nabla_{\perp} \sim a^{-1}$ so that we formally consider low-order poloidal modes. Again, we consider this ordering because it requires the most physics to properly treat. Our equations are also correct in the strongly turbulent regime when ∇_{\perp} is isotropic.

The parallel gradient ∇_{\parallel} has a similar two-scale structure. Toroidal curvature drives pressure and velocity perturbations with $\nabla_{\parallel} \sim R^{-1} \sim \epsilon a^{-1}$. At the same time in a region of order $\Delta \sim \epsilon^2 a$ around the rational surface magnetic shear causes $\mathbf{b} \cdot \nabla \sim k_{\parallel} \sim k \Delta / L_s \sim \Delta / a R \sim \epsilon^3 a^{-1}$, where $ka \sim 1$ and $L_s \sim R$ is the magnetic shear length. The operator $\mathbf{b} \cdot \nabla$ can actually have a range of values between these two limits, depending upon which physical variable it is operating. This two-scale structure is also a characteristic feature of linear resistive modes in toroidal geometry.¹⁶

The ordering of velocities is also more complex than in Strauss' equations. Since to lowest order the plasma is incompressible, $\nabla \cdot \mathbf{v} \sim \nabla_{\perp} \cdot \mathbf{v}_{\perp} \sim 0$. The anisotropic ordering of ∇_{\perp} also implies that \mathbf{v}_{\perp} in the $\nabla \psi$ direction is ϵ^2 smaller than in the $\mathbf{b} \times \nabla \psi$ direction, as indicated in (17f). The magnitude of \mathbf{v}_{\perp} follows from the assumption that convective and time derivatives are comparable,

$$\frac{\partial}{\partial t} \sim \mathbf{v}_{\perp} \cdot \nabla_{\perp}. \quad (18)$$

The scaling of v_{\parallel} and the remaining nonlinear terms in Eqs. (1)–(8) follow entirely from these assumptions. It will be shown, for example, that the pressure fluctuations are of order

$$\tilde{p} \sim \epsilon^2 p_0,$$

so that

$$\nabla_{\perp} p \sim \nabla_{\perp} p_0 \sim \nabla_{\perp} \tilde{p} \sim p_0 / a,$$

where p_0 and \tilde{p} are the "equilibrium" and "perturbed" pressures. Of course, the distinction between "equilibrium" and "perturbed" quantities has no meaning once the system has evolved nonlinearly so that $p_0(\tilde{p})$ actually represents the long (short) space scale component of the pressure. Similar relations hold for the density and other physical quantities.

We now consider the vorticity equation, Eq. (12). Using the ordering presented in (17), we find that the inertia term on the left-hand side of Eq. (12) scales as

$$\frac{1}{B} \frac{1}{\epsilon^2 a} \rho \frac{\epsilon^2}{\tau_A} \epsilon^3 c_A = \frac{\epsilon^4 B}{a^2}.$$

The leading contribution to the curvature term on the right-hand side of Eq. (12) scales as

$$\frac{p_0}{a} \frac{1}{B} \frac{1}{R} = \frac{\beta B}{a R} = \frac{\epsilon^3 B}{a^2},$$

where the main contribution to the curvature comes from the toroidal magnetic field. This term is one order in ϵ larger than the inertia term. However, it tends to be canceled by a portion of the divergence of the parallel current. This cancellation is well known in the linear theory of resistive instabilities.¹⁶ Thus, in order to retain inertia it is necessary to evaluate the two terms on the right-hand side of Eq. (13) including corrections of order ϵ . This requires inclusion of the poloidal curvature and order ϵ corrections to B , \mathbf{b} , and $\mathbf{b} \cdot \nabla$. Since the toroidal and poloidal magnetic fields, B_{ϕ} and B_p , enter $\mathbf{b} \cdot \nabla$ at the same order, ϵ corrections to both B_{ϕ} and B_p must be calculated. From Eq. (9) we see that the pressure and poloidal curvature will only contribute corrections of order ϵ^2 to the magnitude of B . Thus, to first order in ϵ ,

$$B \simeq B_{\phi} \simeq B_0 R_0 / R, \quad (19)$$

where B_0 and R_0 are constants. The magnetic field \mathbf{B} can then be written as

$$\mathbf{B} \simeq \mathbf{B}_p + B_0 R_0 \nabla \phi,$$

where ϕ is the toroidal angle. Since $\nabla \cdot \mathbf{B} = 0$, and $\nabla^2 \phi = 0$, \mathbf{B}_p must be divergence-free so that \mathbf{B} can be written as

$$\mathbf{B} = \nabla \phi \times \nabla \psi + B_0 R_0 \nabla \phi, \quad (20)$$

where ψ is a poloidal flux function. We note, however, that in general field lines do not lie in curves of constant ψ since

$$\mathbf{B} \cdot \nabla \psi = B_0 R_0 \nabla \phi \cdot \nabla \psi \neq 0,$$

the exception being the case of axisymmetry when ψ is independent of ϕ . We note also that if ϵ^2 corrections to B_{ϕ} associated with pressure or poloidal curvature are added to (21), i.e., we include $\mathbf{B}_{\phi 2} = f \nabla \phi \sim \epsilon^2 B_0 R_0 \nabla \phi$, an additional component of the poloidal field \mathbf{B}_{p2} must be added so that

$$\nabla \cdot \mathbf{B} = (\nabla f) \cdot \nabla \phi + \nabla \cdot \mathbf{B}_{p2} = 0.$$

However, the correction B_{p2} is at most of order $a \nabla f \cdot \nabla \phi \sim \epsilon^3 B_0$, which is an ϵ^2 addition to \mathbf{B}_p in Eq. (20) and can be neglected. Thus, the expression for \mathbf{B} in (21) includes all ϵ corrections to both \mathbf{B}_p and B_{ϕ} and is sufficient to evaluate $\mathbf{b} \cdot \nabla$, B , and $\mathbf{b} \times \nabla \kappa$ in Eq. (12).

In order to obtain an equation for ψ , we substitute \mathbf{B} as given in Eq. (20) into Eq. (3) and take the component in the \mathbf{b} direction. Keeping terms to first order in ϵ we find

$$R^2 \nabla \cdot \mathbf{R}^{-2} \nabla \psi = (4\pi/c) R J_{\parallel} = (4\pi/c) R_0 B_0 (J_{\parallel}/B). \quad (21)$$

In evaluating the curvature term in Eq. (12) we must also retain first-order corrections in ϵ . Using Eq. (20) we write

$$\mathbf{b} = \hat{\mathbf{e}}_{\phi} + \mathbf{b}_p, \quad (22)$$

where $\mathbf{b}_p = \mathbf{B}_p/B$. The curvature is then given by

$$\boldsymbol{\kappa} = \hat{\mathbf{e}}_{\phi} \cdot \nabla \hat{\mathbf{e}}_{\phi} + \hat{\mathbf{e}}_{\phi} \cdot \nabla \mathbf{b}_p + \mathbf{b}_p \cdot \nabla \mathbf{b}_p, \quad (23)$$

where the first term is the toroidal field curvature and the second and third terms are smaller than the first by one power of ϵ . The term $\mathbf{b}_p \cdot \nabla \hat{\mathbf{e}}_{\phi}$ does not appear in (23) since it vanishes due to the fact that \mathbf{b}_p lies entirely in the poloidal plane. The quantity which is required for Eq. (12) is $\mathbf{b} \times \boldsymbol{\kappa}$, which, including order ϵ corrections, is given by

$$\mathbf{b} \times \boldsymbol{\kappa} = \hat{\mathbf{e}}_{\phi} \times (\hat{\mathbf{e}}_{\phi} \cdot \nabla \hat{\mathbf{e}}_{\phi}) + \mathbf{b}_p \times (\hat{\mathbf{e}}_{\phi} \cdot \nabla \hat{\mathbf{e}}_{\phi}) + \hat{\mathbf{e}}_{\phi} \times (\mathbf{b}_p \cdot \nabla) \mathbf{b}_p, \quad (24)$$

where the second and third terms are smaller than the first by one power in ϵ . Equation (24) can be further simplified by noting that the second term has no component in the poloidal plane and, since it is a first-order correction, may be dropped. Furthermore, the third term may be rewritten by interchanging the order of the derivative and the cross product. This generates an error of order ϵ^2 , which can be ignored. Thus,

$$\begin{aligned} \mathbf{b} \times \boldsymbol{\kappa} &\approx \hat{\mathbf{e}}_{\phi} \times (\hat{\mathbf{e}}_{\phi} \cdot \nabla \hat{\mathbf{e}}_{\phi}) + \mathbf{b}_p \cdot \nabla [\nabla \psi / (R_0 B_0)] \\ &\approx -\nabla \phi \times \nabla R + \mathbf{b}_p \cdot \nabla [\nabla \psi / (R_0 B_0)]. \end{aligned} \quad (25)$$

We now return to evaluate the velocity. The evolution equation for v_{\parallel} is obtained from (13). To leading order \mathbf{b} is time and space independent so we commute \mathbf{b} with the time derivative to obtain

$$\rho \frac{d}{dt} v_{\parallel} = -\mathbf{b} \cdot \nabla p. \quad (26)$$

We return to Eq. (15) to find $\nabla \cdot \mathbf{v}_{\perp}$. Using the pressure balance relation in Eq. (9), Eq. (15) can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial t} \frac{B^2}{2} + B^2 \left(\nabla \cdot \mathbf{v} - \mathbf{B} \cdot \nabla \frac{v_{\parallel}}{B} + 2\boldsymbol{\kappa} \cdot \mathbf{v} \right) \\ = 4\pi \mathbf{v}_{\perp} \cdot \nabla p - 4\pi \eta J^2 - c^2 \nabla \cdot \eta \nabla p. \end{aligned} \quad (27)$$

From the pressure equation and force balance we have

$$\frac{\partial p}{\partial t} \sim \frac{\partial B^2}{\partial t} \sim \mathbf{v}_{\perp} \cdot \nabla p \sim p \nabla \cdot \mathbf{v}.$$

The first terms on both the left and right sides of (27) are $\beta \sim \epsilon^2$ smaller than $B^2 \nabla \cdot \mathbf{v}$ and can be neglected. The Ohmic heating term ηJ^2 is even smaller so that (27) reduces to

$$\nabla \cdot \mathbf{v}_{\perp} = -2\boldsymbol{\kappa} \cdot \mathbf{v}_{\perp} - (c^2/B_0^2) \nabla \cdot \eta \nabla p. \quad (28)$$

Thus, the curvature of the magnetic field produces a divergence of the perpendicular flow such that

$$\nabla \cdot \mathbf{v}_{\perp} \sim \frac{v_{\perp}}{R} \sim \epsilon^3 \frac{v_{\perp}}{\Delta} \ll \frac{v_{\perp}}{\Delta},$$

so that the fluid motion is nearly incompressible. Finally, we

note that by taking the divergence of our lowest-order velocity in Eq. (14) and using the pressure balance relation to relate ∇B^2 and $\boldsymbol{\kappa} B^2$, we reproduce the first term in (28) so that even the small compressible portion of the flow arises from the electrostatic potential. The velocity v_{\perp} can therefore be written as

$$\mathbf{v}_{\perp} = -c \nabla \times \left(\frac{\mathbf{b} \Phi}{B} \right) + 2c \Phi \frac{\mathbf{b} \times \boldsymbol{\kappa}}{B} - \frac{c^2}{B_0^2} \eta \nabla p,$$

or approximating the incompressible portion to lowest order,

$$\mathbf{v}_{\perp} = \frac{c R_0}{B_0} \nabla \phi \times \nabla \Phi + 2c \Phi \frac{\mathbf{b} \times \boldsymbol{\kappa}}{B} - \frac{c^2}{B_0^2} \eta \nabla p. \quad (29)$$

The term proportional to resistivity in (29) represents classical cross-field particle transport and is formally small in our ordering. However, near marginal stability when $\partial/\partial t$ is smaller than ϵ^2/τ_A , this term can be important so it is retained. With \mathbf{v}_{\perp} given in (29), the evolution equation for p assumes the form

$$\begin{aligned} \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla - \left(\frac{c^2 \gamma_s p}{B_0^2} \right) \nabla \cdot \eta \nabla \right] p \\ + \gamma_s p \left(\mathbf{B} \cdot \nabla \frac{v_{\parallel}}{B} + 2c \frac{\mathbf{b} \times \boldsymbol{\kappa}}{B} \cdot \nabla \Phi \right) \\ = (\gamma_s - 1) \eta (J_{\parallel}^2 + c^2 |\nabla p|^2 / B_0^2). \end{aligned} \quad (30)$$

The magnitude of each of the terms in (30) can now be computed using the ordering in (17). The dominant terms are associated with the divergence of the flow,

$$\gamma_s p \left(\mathbf{B} \cdot \nabla \frac{v_{\parallel}}{B} \right) \sim p \frac{v_{\parallel}}{R} \sim \epsilon^3 \frac{p c_A}{R},$$

$$\gamma_s p 2c \frac{\mathbf{b} \times \boldsymbol{\kappa}}{B} \cdot \nabla \Phi \sim p \boldsymbol{\kappa} \cdot \mathbf{v} \sim p \frac{v_{\perp}}{R} \sim \epsilon^3 \frac{p c_A}{R},$$

which are one order in ϵ larger than dp/dt ,

$$\frac{\partial p}{\partial t} \sim \mathbf{v}_{\perp} \cdot \nabla p \sim \epsilon^5 \frac{c_A p}{a} \sim \epsilon^4 \frac{p c_A}{R},$$

and therefore must balance to lowest order. The divergence of the perpendicular velocity drives a parallel flow with $v_{\parallel} \sim v_{\perp}$. This result is a consequence of our assumption $\partial/\partial t \sim \epsilon^2 c_A/R \sim \epsilon c_s/R \ll c_s/R$, which also implies that the parallel pressure gradient, which is driven by the curvature, is flattened by the sound wave. The residual parallel pressure gradient is sufficient to drive the parallel flow. The lowest-order cancellation of $\nabla \cdot \mathbf{v}_{\perp}$ and $\mathbf{B} \cdot \nabla (v_{\parallel}/B)$ implies, as in the momentum equation, that the poloidal curvature and order ϵ corrections to \mathbf{B}_p and B_{ϕ} must be retained in Eq. (30). The Ohmic heating terms on the right side of (30) are formally small but are retained to conserve energy.

Finally, the inertia term in the momentum equation [Eq. (12)] can be expressed as

$$\frac{\mathbf{b} \cdot \nabla \times \rho}{B} \frac{d\mathbf{v}}{dt} \sim \nabla \cdot \frac{\rho}{B_0^2} \frac{d}{dt} \nabla \Phi,$$

where we have approximated \mathbf{B} , \mathbf{b} , and \mathbf{v} to lowest order since the inertia is already one order smaller than the indi-

vidual terms on the right side of (12). Our equation for the potential Φ now becomes

$$\frac{c^2}{B_0^2} \nabla \rho \frac{d}{dt} \nabla \Phi = \mathbf{B} \cdot \nabla \frac{J_{\parallel}}{B} + 2c \frac{\mathbf{b} \times \boldsymbol{\kappa}}{B} \cdot \nabla p. \quad (31)$$

To complete our parallel Ohm's law in Eq. (16), we calculate \mathbf{E} by inserting Eq. (20) for \mathbf{B} into Eq. (5),

$$\mathbf{E} = -\nabla \Phi + \frac{1}{c} \frac{\partial \psi}{\partial t} \nabla \phi.$$

Equation (16) then yields an evolution equation for the poloidal flux function ψ ,

$$\frac{1}{cR} \frac{\partial \psi}{\partial t} = \mathbf{b} \cdot \nabla \Phi + \eta J_{\parallel}. \quad (32)$$

We now summarize our system of equations. Five equations which evolve the quantities Φ , v_{\parallel} , ψ , p , and ρ are the vorticity equation,

$$\frac{c^2}{B_0^2} \nabla \rho \frac{d}{dt} \nabla \Phi = \mathbf{B} \cdot \nabla \frac{J_{\parallel}}{B} + 2c \frac{\mathbf{b} \times \boldsymbol{\kappa}}{B} \cdot \nabla p; \quad (33)$$

the parallel momentum equation,

$$\rho \frac{dv_{\parallel}}{dt} = -\mathbf{b} \cdot \nabla p; \quad (34)$$

the parallel component of Ohm's law,

$$\frac{1}{cR} \frac{\partial \psi}{\partial t} = \mathbf{b} \cdot \nabla \Phi + \eta J_{\parallel}; \quad (35)$$

the pressure equation,

$$\frac{dp}{dt} + \gamma_s p \left(\nabla \cdot \mathbf{v}_{\perp} + \mathbf{B} \cdot \nabla \frac{v_{\parallel}}{B} \right) = (\gamma_s - 1) \eta \left[\left(\frac{c \nabla p}{B_0} \right)^2 + J_{\parallel}^2 \right]; \quad (36)$$

and conservation of mass density,

$$\frac{d\rho}{dt} + \rho \left(\nabla \cdot \mathbf{v}_{\perp} + \mathbf{B} \cdot \nabla \frac{v_{\parallel}}{B} \right) = 0. \quad (37)$$

The quantities \mathbf{B} , b , \mathbf{v}_{\perp} , J_{\parallel} , and $\mathbf{b} \times \boldsymbol{\kappa}$ are given in terms of Φ , ψ , and p . The magnetic field is represented as

$$\mathbf{B} = \nabla \phi \times \nabla \psi + R_0 B_0 \nabla \phi, \quad (38)$$

where ϕ is the toroidal angle, R_0 is the major radius of the magnetic axis, $R^{-1} = |\nabla \phi|$, and B_0 is a constant. The modulus of \mathbf{B} is given by

$$B = R_0 B_0 / R, \quad (39)$$

and

$$\mathbf{b} = \mathbf{B} / B.$$

The quantity $\mathbf{b} \times \boldsymbol{\kappa}$ is

$$\mathbf{b} \times \boldsymbol{\kappa} = -\nabla \phi \times \nabla R + \mathbf{b} \cdot \nabla [\nabla \psi / (R_0 B_0)]. \quad (40)$$

The perpendicular component of the fluid velocity is expressed in terms of Φ and p ,

$$\mathbf{v}_{\perp} = -\frac{cR_0 \nabla \Phi \times \nabla \phi}{B_0} + 2c\Phi \frac{\mathbf{b} \times \boldsymbol{\kappa}}{B} - \frac{c^2 \eta \nabla p}{B_0^2}, \quad (41)$$

and the total time derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_{\perp} \cdot \nabla + v_{\parallel} \mathbf{b} \cdot \nabla. \quad (42)$$

Finally, the parallel current satisfies

$$R^2 \nabla \cdot \mathbf{R}^{-2} \nabla \psi = (4\pi/c) R_0 B_0 (J_{\parallel} / B). \quad (43)$$

Our equations reduce to the high- β equations of Strauss⁶ if the curvature is evaluated to lowest order [second term in (40) is neglected], the plasma flow is taken to be incompressible [v_{\parallel} is neglected and only the first term in (41) is retained], the mass density ρ is taken to be a constant, and R is approximated by R_0 in (35) and (41). With this system of equations, Strauss showed that the total energy

$$\mathcal{W}_s = \int d\mathbf{x} \left(\frac{1}{2} \rho c^2 \left| \frac{\nabla \Phi}{B_0} \right|^2 + \frac{1}{8\pi} \left| \frac{\nabla \psi}{R_0} \right|^2 + 2p(R - R_0) \right)$$

is conserved. The first two terms are the flow and magnetic energy, respectively, while the last term represents the potential energy of an incompressible plasma in the "gravitational" potential $2(R - R_0)$. In contrast, the set of equations given in (33)–(43) exactly conserves the energy

$$\begin{aligned} W = \int d\mathbf{x} \left[\frac{1}{8\pi} \left| \frac{\nabla \psi}{R} \right|^2 + \frac{p}{\gamma_s - 1} \right. \\ \left. + \frac{1}{2} \rho \left(v_{\parallel}^2 + \frac{c^2}{B_0^2} |\nabla \Phi|^2 \right) \right], \quad (44) \end{aligned}$$

and in our ordering scheme the time variation of each of the individual terms in (44) are comparable. To ensure total energy conservation, certain symmetries in the reduced equations must be maintained. For example, in the convective derivatives $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ in Eqs. (3) and (20)–(22), the component of $\mathbf{v} \cdot \nabla$ arising from the compressible part of \mathbf{v} is formally small in our ordering yet it must be included to conserve energy.

Frequently the mass density ρ is not evolved in MHD simulations. However, simply replacing ρ in (33) and (34) by a constant and discarding (37) does not yield a set of energy-conserving equations. The time variation of ρ and the compressibility of the flow are inherently linked so that in replacing ρ by a constant the velocities in the convective derivatives in (33) and (34) must be approximated by their incompressible counterparts.

The average magnetic well in a tokamak with $q > 1$ has a strong stabilizing influence on pressure-driven modes. It is shown in the Appendix that by linearizing our reduced equations we can reproduce the Mercier criterion in the large-aspect-ratio, low- β limit,¹²

$$D_M = \frac{2r}{R_0^2} \frac{L_s^2}{B_0^2} \frac{dp}{dr} \left(1 - \frac{1}{q^2(r)} \right) < \frac{1}{4}, \quad (45)$$

where $L_s = q^2 R_0 / (r dq/dr)$ is the magnetic shear length.

The system of equations derived here for resistive modes in a low- β plasma contains as a special limit the high- β equations of Strauss.⁶ This does not imply that our equations are valid for resistive modes in a high- β plasma. Rather, they are valid for ideal modes in a high- β plasma (as are Strauss' equations) and resistive modes in a low- β plasma.

III. NONLINEAR REDUCED BRAGINSKII EQUATIONS

The resistive MHD equations are a useful starting point for studying resistive modes in toroidal plasmas. However, many physical effects are not included in these equations.

The rotation of electrostatic or electromagnetic perturbations at the diamagnetic frequency

$$\omega_* \sim \frac{k_\theta c}{eBn} \frac{dp}{dr} \sim \rho_s \frac{c_s}{a^2} \quad (46)$$

has been observed in many experiments and has been predicted to have a strong stabilizing influence on both current¹⁴ and pressure-driven modes.¹⁷ In the definition of ω_* , n is the plasma density and $\rho_s = c_s/\Omega_i$ is the ion Larmor radius based on the sound speed c_s . To incorporate the diamagnetic rotation into the ordering presented in Sec. II, we require

$$\frac{\partial}{\partial t} \sim \frac{\epsilon^2}{c_A} \sim \frac{\rho_s c_s}{a^2},$$

or

$$\rho_s/a \sim \epsilon^2.$$

In resistive MHD, $\rho_s/a \rightarrow 0$ so these effects are not included.

In recent years the importance of transport induced by magnetic fluctuations in tokamak discharges has been widely acknowledged both during major disruptions³ and in more quiescent phases of the discharge.^{2,18} The transport resulting from radial magnetic fluctuations \tilde{B}_r becomes comparable to that associated with radial flow \tilde{v}_r when

$$\left(\frac{\tilde{B}_r}{B}\right)^2 \frac{v_e^2}{v_{ei}^2} \sim \tilde{v}_r^2 \Delta t,$$

where $v_e = (2T_e/m_e)^{1/2}$ is the electron thermal velocity, $v_{ei} = (4/3)(2\pi)^{1/2} n e^4 \ln \Lambda / m_e^{1/2} T_e^{3/2}$ is the electron-ion collision frequency, and Δt is the correlation time of the radial flow. This relation can be rewritten as

$$\frac{\tau_r}{\tau_A} \left(\frac{\rho_s}{a}\right)^2 \left(\frac{\tilde{B}_r}{B}\right)^2 \sim \left(\frac{\tilde{v}_r}{c_A}\right)^2 \frac{\Delta t}{\tau_A}.$$

Our ordering in Sec. II requires $\tilde{B}_r/B \sim \tilde{v}_r/c_A$ and $\Delta t/\tau_A \sim \epsilon^2$ so that transport induced by magnetic fluctuations again becomes important when

$$\rho_s/a \sim \epsilon^2,$$

and so is not included in the MHD equations. In the past such transport has been incorporated into MHD simulations of resistive instabilities non-self-consistently² and hence the results must be considered as tentative.

To derive a set of nonlinear equations which incorporate both diamagnetic rotation and other effects, we begin with the Braginskii fluid equations.¹¹ In the limit

$$\begin{aligned} \rho_e^2 \nabla_\perp^2 &\ll 1, \\ \frac{\partial}{\partial t} &\ll v_{ei}, \\ \frac{v_e^2}{v_{ei}^2} (\mathbf{b} \cdot \nabla)^2 &\ll 1, \end{aligned} \quad (47)$$

the electron inertia and the stress tensor in the electron momentum equation can be neglected and the stress tensor can be discarded in the electron temperature equation. The electron equations can then be written as

$$\frac{Dn_e}{Dt} + n_e \nabla \cdot \mathbf{v}_e = 0, \quad (48)$$

$$\mathbf{R}_{ei} = n_e e [\mathbf{E} + (\mathbf{v}_e \times \mathbf{B})/c] + \nabla p_e, \quad (49)$$

$$\frac{3}{2} n_e \frac{DT_e}{Dt} + n_e T_e \nabla \cdot \mathbf{v}_e + \nabla \cdot \mathbf{q}_e = \mathbf{R}_{ei} \cdot \frac{\mathbf{J}}{ne}, \quad (50)$$

where

$$\mathbf{R}_{ei} = n_e e \eta \cdot \mathbf{J} - 0.71 n_e \mathbf{b} \mathbf{b} \cdot \nabla T_e - \frac{3}{2} (n_e v_{ei}/\Omega_e) \mathbf{b} \times \nabla T_e \quad (51)$$

is the momentum transfer between electrons and ions with

$$\eta = (\mathbf{I} - \mathbf{b} \mathbf{b}) \eta_\perp + \mathbf{b} \mathbf{b} \eta_\parallel,$$

and

$$\eta_\parallel = 0.5 \eta_\perp = 0.5 m_e v_{ei} / n_e e^2.$$

The first term in Eq. (51) is the usual collisional momentum transfer due to the relative mean velocities of electrons and ions. The second and third terms are the so-called parallel and perpendicular thermal forces. The quantity \mathbf{q}_e in Eq. (50) is the heat flux

$$\begin{aligned} \mathbf{q}_e = & -0.71 T_e \frac{J_\parallel}{e} \mathbf{b} + \frac{3}{4} \rho_e^2 v_{ei} \frac{\mathbf{J} \times \mathbf{B}}{c} - 2.4 n_e \rho_e^2 v_{ei} \nabla_\perp T_e \\ & - \frac{5}{2} \frac{n_e T_e}{m_e \Omega_e} \mathbf{b} \times \nabla T_e - \kappa_\parallel \mathbf{b} \mathbf{b} \cdot \nabla T_e, \end{aligned} \quad (52)$$

with $\rho_e = v_e/\Omega_e$ the electron Larmor radius, and $\kappa_\parallel = 3.2 n_e T_e / m_e v_{ei}$ the parallel thermal conductivity. The first two terms in Eq. (52) are heat fluxes driven by the current and are conjugate to the thermal force terms in Eq. (51). The third and fifth terms are due to classical perpendicular and parallel thermal conductivity, and the fourth term is the collisionless gyro heat flux. In Eqs. (48) and (50) the total time derivative of the electron fluid is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla. \quad (53)$$

The ions are taken to be cold so we have only the ion continuity and momentum equations

$$\frac{dn_i}{dt} + n_i \nabla \cdot \mathbf{v}_i = 0, \quad (54)$$

$$m_i n_i \frac{d\mathbf{v}_i}{dt} = n_i e \left(\mathbf{E} + \frac{\mathbf{v}_i \times \mathbf{B}}{c} \right) - \mathbf{R}_{ei}. \quad (55)$$

The limitations of these equations must be considered in applying them to tokamak discharges. Of the three inequalities in (47), the first is always satisfied for modes of interest, the second is satisfied for low poloidal and toroidal mode numbers even for reactor parameters, while the third is almost never satisfied. In toroidal geometry $\mathbf{b} \cdot \nabla \sim R^{-1}$ so that this inequality requires

$$v_e/v_{ei} \ll R,$$

i.e., that there are no trapped particles. A second limitation is the neglect of ion temperature. For values of the resistivity in present tokamak experiments, the scale size Δ of unstable current- or pressure-driven modes (in linear theory) is of order ρ_i . Effects of finite ion Larmor radius can therefore only be properly incorporated into the analysis by using kinetic theory. Such a calculation is beyond the scope of the present investigation so we consider the limit $T_i = 0$ as a model. It should also be noted that there is some evidence that finite

ion temperature does not strongly modify the stability of dissipative modes.¹⁹

With these limitations in mind, we proceed to develop a set of reduced equations from Eqs. (48)–(55). The equations can be made structurally more similar to the resistive MHD equations by using the electron force balance equation in (49) to eliminate \mathbf{R}_{ei} in Eq. (55) and then noting that because $m_e/m_i \ll 1$, the fluid velocity $v \sim v_i$. Equation (55) then becomes

$$m_i n \frac{d\mathbf{v}}{dt} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p, \quad (56)$$

when $n = n_e = n_i$ because of charge neutrality and $p = p_e$ is the electron pressure. Equation (56) is identical to its MHD counterpart in Eq. (1). The ion continuity equation can similarly be rewritten as

$$\frac{dn}{dt} + n \nabla \cdot \mathbf{v} = 0, \quad (57)$$

which is the Braginskii counterpart of the equation for the mass density in MHD. The equation for electron force balance in (49) combined with Eq. (51) for the momentum transfer constitute Ohm's law for the Braginskii equations. This Ohm's law can also be transformed to appear more like its MHD counterpart by eliminating \mathbf{v}_e in favor of \mathbf{v} and \mathbf{J} ,

$$\mathbf{v}_e = \mathbf{v} - \mathbf{J}/ne,$$

and then using (56) to obtain

$$\begin{aligned} \eta \cdot \mathbf{J} = \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} - \frac{m_i}{e} \frac{d\mathbf{v}}{dt} + 0.71 \frac{\mathbf{b} \cdot \nabla T}{e} \\ + \frac{3}{2} \frac{v_{ei}}{e \Omega_e} \mathbf{b} \times \nabla T. \end{aligned} \quad (58)$$

The anisotropic resistivity, ion inertia (Hall term), and the two thermal force terms are new effects which did not appear in resistive MHD.

Finally, the electron velocity \mathbf{v}_e can also be eliminated from Eq. (50) for T_e which can then be rewritten as

$$\begin{aligned} \frac{3}{2} n \frac{dT}{dt} - \frac{3}{2} \left(1 + \frac{5}{6} \beta\right) \frac{J_{\parallel}}{e} \mathbf{b} \cdot \nabla T - \frac{5}{2} n \rho_e v_e \mathbf{b} \times \kappa \cdot \nabla T \\ - \frac{5}{8} \rho_e v_e \beta \nabla n \times \mathbf{b} \cdot \nabla T + \frac{3}{2} \frac{n}{\Omega_i} \left(1 + \frac{5}{6} \beta\right) \\ \times \frac{d\mathbf{v}}{dt} \times \mathbf{b} \cdot \nabla T - \mathbf{B} \cdot \nabla \frac{\kappa_{\parallel}}{B^2} \mathbf{B} \cdot \nabla T - 1.7 \nabla_{\perp} \cdot n \rho_e^2 v_{ei} \nabla_{\perp} T \\ + \frac{3}{4} \frac{\rho_e^2 v_{ei}}{cT} \mathbf{J} \times \mathbf{B} \cdot \nabla T \\ = -nT \nabla \cdot \mathbf{v} - \frac{T}{ne} J_{\parallel} \mathbf{b} \cdot \nabla n - 0.71 \frac{T}{e} \mathbf{B} \cdot \nabla \frac{J_{\parallel}}{B} \\ - \frac{T}{\Omega_i} \frac{d\mathbf{v}}{dt} \times \mathbf{b} \cdot \nabla n - \frac{3}{4} \nabla \cdot \rho_e^2 v_{ei} T \nabla n + \mathbf{J} \cdot \eta \cdot \mathbf{J}. \end{aligned} \quad (59)$$

Most of the terms in Eq. (59) scale as ρ_s/a . For example,

$$\frac{T}{e} \mathbf{B} \cdot \nabla \frac{J_{\parallel}}{B} \sim \frac{T}{e} \frac{1}{R} \frac{c B_p}{4\pi a} \sim \frac{nT}{\tau_A} \epsilon \frac{\rho_s}{a},$$

so that in the limit $\rho_s/a \rightarrow 0$, Eq. (59) simplifies to

$$\begin{aligned} \frac{3}{2} \frac{dp}{dt} - 1.7 \nabla_{\perp} \cdot n \rho_e^2 v_{ei} \nabla_{\perp} T + \frac{3}{4} \frac{\rho_e^2 v_{ei}}{cT} \mathbf{J} \times \mathbf{B} \cdot \nabla T \\ = -\frac{5}{2} p \nabla \cdot \mathbf{v} - \frac{3}{4} \nabla \cdot \rho_e^2 v_{ei} T \nabla n + \mathbf{J} \cdot \eta \cdot \mathbf{J}, \end{aligned} \quad (60)$$

where the ion continuity equation has been used to write (60) as an equation for p . Aside from the extra cross-field transport terms and the anisotropic resistivity, Eq. (60) for p is identical to Eq. (6) with $\gamma_s = \frac{5}{3}$.

The basic operations which were carried out to obtain a set of reduced equations in the MHD limit can now be repeated on Eqs. (56)–(60). The parallel Ohm's law, obtained by dotting Eq. (58) with \mathbf{b} and using (56) to eliminate \mathbf{v} ,

$$\frac{1}{cR} \frac{\partial \psi}{\partial t} = \eta_{\parallel} J_{\parallel} + \mathbf{b} \cdot \nabla \Phi - \frac{1}{ne} \mathbf{b} \cdot \nabla p - \frac{0.71}{e} \mathbf{b} \cdot \nabla T, \quad (61)$$

now contains pressure terms which cause magnetic perturbations to rotate at the diamagnetic frequency.^{13,14} The perpendicular fluid velocity \mathbf{v}_{\perp} now includes a correction associated with the ion polarization drift

$$\begin{aligned} \mathbf{v}_{\perp} = -\frac{cR_0}{B_0} \nabla \Phi \times \nabla \phi + 2c\Phi \frac{\mathbf{b} \times \kappa}{B} \\ - \frac{c}{B_0 \Omega_{i0}} \frac{d}{dt} \nabla \Phi - \frac{\eta_{\perp} c^2}{B_0^2} \left(T \nabla_{\perp} n - \frac{1}{2} n \nabla_{\perp} T \right). \end{aligned} \quad (62)$$

From Eq. (59), the electron temperature satisfies

$$\begin{aligned} \frac{3}{2} n \frac{dT}{dt_0} - 5 \frac{ncT}{e} \nabla \cdot \left(\frac{\mathbf{b} \times \kappa}{B} T \right) - \frac{3}{2} \frac{J_{\parallel}}{e} \mathbf{b} \cdot \nabla T \\ - \mathbf{B} \cdot \nabla \frac{\kappa_{\parallel}}{B^2} \mathbf{B} \cdot \nabla T - 2.4 \nabla \cdot (n \rho_e^2 v_{ei}) \nabla T \\ = -nT \nabla \cdot \mathbf{v}_0 + 1.71 \frac{T}{e} \mathbf{B} \cdot \nabla \frac{J_{\parallel}}{B} - \frac{T}{ne} J_{\parallel} \mathbf{b} \cdot \nabla n \\ + \rho_e v_e \mathbf{b} \times \kappa \cdot \nabla p + \frac{\eta_{\perp} c^2}{B_0^2} |\nabla p|^2 \\ - \frac{3}{4} \frac{1}{T} \nabla \cdot T \rho_e^2 v_{ei} \nabla p + \eta_{\parallel} J_{\parallel}^2, \end{aligned} \quad (63)$$

where

$$\mathbf{v}_0 = \mathbf{v} + \frac{c}{B_0 \Omega_{i0}} \frac{d}{dt} \nabla \Phi, \quad (64)$$

$$\frac{d}{dt_0} = \frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla.$$

As in the case of the resistive MHD equations, the cross-field transport terms in Eqs. (62) and (63) are formally small except near marginal stability. The reduced nonlinear Braginskii fluid equations for toroidal plasmas are virtually identical to those derived from resistive MHD. Only the evolution equations for ψ and p in Eqs. (35) and (36) and the fluid velocity \mathbf{v} in Eq. (41) are altered. They are replaced by the corresponding equations for ψ , T , and \mathbf{v} in Eqs. (61)–(63).

The nonlinear reduced Braginskii equations satisfy the same energy conservation law as do the reduced resistive MHD equations. The energy \mathcal{W} defined in Eq. (44) is conserved exactly by the reduced Braginskii equations for $\gamma_s = \frac{5}{3}$. The parallel pressure-driven currents, which are described by the Ohm's law in Eq. (61) and the complicated

transport and compression terms in the temperature equation in (63), ultimately cause a redistribution of thermal energy but introduce no new sources or sinks.

The influence of diamagnetic rotation on the nonlinear evolution of resistive tearing modes in cylindrical geometry was studied previously by assuming T was constant in both space and time, neglecting v_{\parallel} and the magnetic curvature.^{20,21} In this limit the reduced Braginskii fluid equations are given by

$$\begin{aligned} \frac{c^2}{B_0^2} \nabla \cdot \rho \frac{d}{dt} \nabla \Phi &= \mathbf{b} \cdot \nabla J_{\parallel}, \\ \nabla^2 \psi &= 4\pi J_{\parallel} / c, \\ \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} &= 0, \\ \mathbf{B} &= \hat{z} \times \nabla \psi + \hat{z} B_0, \\ \mathbf{v} &= \frac{c}{B_0} \hat{z} \times \nabla \Phi - \frac{c}{B_0 \Omega_{i0}} \frac{d}{dt} \nabla \Phi - \frac{\eta_{\perp} c^2}{B_0^2} T \nabla_{\perp} n, \\ \frac{1}{c} \frac{\partial \psi}{\partial t} &= \eta_{\parallel} J_{\parallel} + \mathbf{b} \cdot \nabla \Phi - \frac{T}{ne} \mathbf{b} \cdot \nabla n, \end{aligned} \quad (65)$$

where the major radius R has been absorbed into the definition of ψ and $\mathbf{b} = \mathbf{B}/B_0$. These equations are also identical to those recently derived by Hazeltine.²² Since the temperature has been taken as a constant in these equations, the usual energy conservation law given in Eq. (44) should no longer be satisfied. Nevertheless, multiplying the vorticity equation by Φ and integrating over \mathbf{x} , we calculate

$$\begin{aligned} \frac{\partial}{\partial t} \int d\mathbf{x} \left(\frac{1}{2} \rho \frac{c^2}{B_0^2} |\nabla \Phi|^2 + \frac{1}{8\pi} |\nabla \psi|^2 + p \ln(n) \right) \\ = - \int d\mathbf{x} \left(\eta_{\parallel} J_{\parallel}^2 + \eta_{\perp} \frac{c^2}{B_0^2} |\nabla_{\perp} n|^2 \right). \end{aligned} \quad (66)$$

The right side of the equation is the rate of energy loss due to the dissipation of the parallel and perpendicular currents. This energy is lost since T is taken to be a constant. The first and second terms on the left side of Eq. (66) are the usual expressions for the perpendicular flow and magnetic energy, respectively. The third term on the left side of Eq. (66) plays the role of the internal energy.

IV. CONCLUSIONS

We have derived two systems of nonlinear equations which describe resistive tearing and ballooning equations in large-aspect-ratio, low- β tokamaks. The first system of equations is based on the one-fluid resistive MHD equations and the second system is based on the two-fluid Braginskii equations. The compressional Alfvén wave has been eliminated from both sets of equations.

The system of equations based on resistive MHD is an extension of the original work of Strauss.^{5,6} Strauss' reduced equations were derived by expanding the ideal MHD equations in a power series in the inverse aspect ratio ϵ with an ordering based on global ideal modes. We have developed an ordering appropriate for resistive modes, which grow more slowly than ideal modes and have localized radial structure near mode rational surfaces. Our equations included the sta-

bilizing effects of plasma compressibility and average favorable curvature.

Recently, Schmalz²³ and Strauss²⁴ have derived equations to describe resistive modes in a large-aspect-ratio tokamak. Their basic approach was to simply retain Strauss' original ordering^{5,6} for the time and space scales and expand the MHD equations to higher order in powers of ϵ . In principle, this procedure will eventually yield a set of equations which are sufficiently accurate to describe resistive modes in a large-aspect-ratio tokamak, yet this approach is somewhat arbitrary. If the extra terms are higher order in ϵ , why must they be retained? The answer is, as we have emphasized in the Introduction, that the original ordering developed by Strauss is not appropriate for resistive modes. Nevertheless, it is useful to compare our results with those obtained by this other approach. Schmalz neglected both the parallel compression and cross-field particle transport in evaluating $\nabla \cdot \mathbf{v}$ in the equation for pressure p . In this limit the compressible stabilization of low mode number resistive ballooning mode is absent²⁵ and his equations are therefore not appropriate for the study of these resistive curvature-driven modes. Strauss' equations are essentially the same as ours in the limit $\beta \sim \epsilon^2$ (he retains the diamagnetic depression of B_{ϕ} for $\beta \sim \epsilon$) although he also neglects cross-field particle transport. Because we have not included the diamagnetic depression of B , our equations cannot be used to reproduce the stabilization of the ideal $m = 1$ mode at low β .^{26,27}

The second system of equations, which is based on the two-fluid Braginskii equations, is appropriate when the time scale is comparable to the diamagnetic drift frequency. This system of equations includes the effects of compressibility and average favorable curvature, as well as the effects of parallel electron pressure gradients in Ohm's law, the ion polarization drift, and parallel electron thermal conductivity. These latter effects have been shown to have a strong stabilizing influence on linear tearing modes.^{13,14} In addition, since parallel thermal conduction is included in the equation for the electron temperature, these equations are appropriate for self-consistently determining whether an anomalously large radial electron heat flux can result from the parallel transport of heat along the braided magnetic fields produced by resistive tearing and ballooning modes.

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APPENDIX: LINEARIZED EQUATIONS AND THE MERCIER CRITERION

We now show that our reduced resistive MHD equations are sufficiently accurate to reproduce the Mercier criterion for a large-aspect-ratio torus. To show this, we first linearize the equations about an axisymmetric equilibrium and look for solutions in a region $\Delta \ll a$ around the rational surface $q = m/n$. The integers m and n are poloidal and toroidal mode numbers and the safety factor $q(\psi)$, which is a

function only of the equilibrium flux ψ , is defined as

$$q(\psi) = \langle \mathbf{B} \cdot \nabla \phi \rangle / \langle \mathbf{B} \cdot \nabla \chi \rangle,$$

where ϕ and χ are the toroidal and poloidal angles, respectively, and the field line average of a quantity A is given by

$$\langle A \rangle \equiv \frac{\int (dl/B) A}{\int dl/B}.$$

It is convenient to expand the perturbed quantities in Hamada coordinates as

$$\tilde{A}(\theta, \psi) \exp(\gamma t + i n u),$$

where γ is the growth rate, $u = \zeta - m\theta/n$, and θ and ζ are the poloidal and toroidal Hamada coordinates. Explicit representations for θ and ζ can be derived from the equations

$$\mathbf{B} \cdot \nabla \theta = \langle \mathbf{B} \cdot \nabla \chi \rangle,$$

$$\mathbf{B} \cdot \nabla \zeta = \langle \mathbf{B} \cdot \nabla \phi \rangle,$$

so that θ and ζ represent the field line averaged poloidal and toroidal angles. The safety factor q is given by the local pitch of θ and ζ ,

$$q(\psi) = \langle \mathbf{B} \cdot \nabla \zeta \rangle / \langle \mathbf{B} \cdot \nabla \theta \rangle.$$

These coordinates are useful because \mathbf{B} and $\mathbf{B} \cdot \nabla$ have simple representations:

$$\mathbf{B} = \nabla u \times \nabla \psi + (q - m/n) \nabla \psi \times \nabla \theta,$$

$$\mathbf{B} \cdot \nabla \simeq \frac{B_0}{qR_0} \left[\frac{\partial}{\partial \theta} + \left(q - \frac{m}{n} \right) \frac{\partial}{\partial u} \right].$$

Our linearized equations are given by

$$\gamma \frac{4\pi\rho}{B_0^2} |\nabla\psi|^2 \tilde{\Phi}_{\psi\psi} = \mathbf{B} \cdot \nabla \frac{\tilde{J}_{\parallel}}{B} + 8\pi \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \tilde{p}}{B}, \quad (\text{A1})$$

$$|\nabla\psi|^2 \tilde{A}_{\psi\psi} = \tilde{J}_{\parallel} B / B_0, \quad (\text{A2})$$

$$\gamma \tilde{p} + i n p_{\psi} \tilde{\Phi} + \gamma_s p \left(2 \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \tilde{\Phi}}{B} + \mathbf{B} \cdot \nabla \frac{\tilde{v}_{\parallel}}{B} - \frac{4\pi}{B_0^2} D |\nabla\psi|^2 \tilde{p}_{\psi\psi} \right) = 0, \quad (\text{A3})$$

$$\gamma B \rho \tilde{v}_{\parallel} = - \mathbf{B} \cdot \nabla \tilde{p} - i n B_0 p_{\psi} \tilde{A}, \quad (\text{A4})$$

$$D B \tilde{J}_{\parallel} = - \mathbf{B} \cdot \nabla \tilde{\Phi} + \gamma B_0 \tilde{A}, \quad (\text{A5})$$

where \tilde{v}_{\parallel} and \tilde{p} are the perturbed velocity and pressure, $c\tilde{J}_{\parallel}/4\pi$ is the perturbed parallel current, $\tilde{\Phi}/c$ is the perturbed electrostatic potential, and $B_0^2 R_0 \tilde{A}/B^2$ is the perturbed poloidal flux function. The quantity D is the resistive diffusion coefficient

$$D = c^2 \eta / 4\pi,$$

and the subscript ψ denotes $\partial/\partial\psi$.

In deriving Eqs. (A1)–(A5) we have assumed that the gradient operator when acting on perturbed quantities is dominated by the derivative in the ψ direction, i.e., $\nabla \simeq |\nabla\psi|^{-1} \nabla\psi \cdot \nabla \sim \Delta^{-1} \gg a^{-1}$. Furthermore, terms resulting from gradients of the equilibrium current density are dropped because they are not important in the vicinity of the rational surface.

As in the case of arbitrary aspect ratio and arbitrary β , we can further simplify Eqs. (A1)–(A5) by deriving a set of equations for the field-line-averaged (θ -averaged) quantities, $\langle \tilde{\phi} \rangle$, $\langle \tilde{A} \rangle$, $\langle \tilde{p} \rangle$, and $\langle \tilde{v}_{\parallel} \rangle$. We carry out this calculation in

two steps. In the first step we define a small parameter

$$\bar{\epsilon} \equiv \Delta / a,$$

which is a measure of the characteristic length of the perturbations along $\nabla\psi$, and scale the growth rate and resistivity as

$$\gamma \sim \Delta \sim \bar{\epsilon},$$

$$\eta \sim \Delta^3 \sim \epsilon^{-3}.$$

This ordering yields the familiar scaling $\gamma \sim \eta^{1/3}$ for resistive modes and reduces to our previous ordering for $\bar{\epsilon} \sim \epsilon^2$. Nevertheless, at this stage we consider $\bar{\epsilon}$ to be independent of the aspect ratio ϵ and derive the averaged equations by carrying out an expansion in $\bar{\epsilon}$, taking all equilibrium quantities and their variations to be of order unity. In the second step, we expand the coefficients in the averaged equations in powers of ϵ , the aspect ratio, to obtain the Mercier criterion, for a large-aspect-ratio, low- β tokamak.

The operator $\mathbf{B} \cdot \nabla$ can be written as

$$\mathbf{B} \cdot \nabla = \mathbf{B} \cdot \nabla^0 + \mathbf{B} \cdot \nabla^1,$$

where

$$\mathbf{B} \cdot \nabla^0 = \frac{B_0}{qR_0} \frac{\partial}{\partial \theta},$$

and

$$\mathbf{B} \cdot \nabla^1 = \frac{B_0}{qR_0} \left(q - \frac{m}{n} \right) \frac{\partial}{\partial u} = \frac{iB_0}{qR_0} (nq - m),$$

with

$$\mathbf{B} \cdot \nabla^1 \sim \bar{\epsilon} \mathbf{B} \cdot \nabla^0.$$

Accordingly, we expand the variables $\tilde{\phi}$, \tilde{p} , \tilde{J}_{\parallel} , \tilde{A}_{\parallel} , and \tilde{v}_{\parallel} in an ascending series in ϵ . With our assumptions about the scaling of the growth rate, the resistivity, and the perpendicular and parallel derivatives, we find that Eqs. (A1)–(A5) require

$$\tilde{\Phi} = \tilde{\Phi}^0 + \tilde{\Phi}^1 + \dots,$$

$$\tilde{p} = \tilde{p}^{-1} + \tilde{p}^0 + \dots,$$

$$\tilde{A} = \tilde{A}^0 + \tilde{A}^1 + \dots,$$

$$\tilde{J}_{\parallel} = \tilde{J}_{\parallel}^{-2} + \tilde{J}_{\parallel}^{-1} + \dots,$$

and

$$\tilde{v}_{\parallel} = \tilde{v}_{\parallel}^{-1} + \tilde{v}_{\parallel}^0 + \dots,$$

where the superscripts refer to the order in $\bar{\epsilon}$ of each quantity. In this case the leading-order versions of Eqs. (A1)–(A5) become

$$0 = \mathbf{B} \cdot \nabla \frac{\tilde{J}_{\parallel}^{-2}}{B} + 8\pi \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \psi}{B} \tilde{p}^{-1}, \quad (\text{A6})$$

$$|\nabla\psi|^2 \tilde{A}_{\psi\psi}^0 = \tilde{J}_{\parallel}^{-2} B / B_0, \quad (\text{A7})$$

$$2 \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \psi}{B} \tilde{\Phi}_{\psi}^0 + \mathbf{B} \cdot \nabla^0 \frac{\tilde{v}_{\parallel}^{-1}}{B} = 0, \quad (\text{A8})$$

$$0 = - \mathbf{B} \cdot \nabla^0 \tilde{p}^{-1}, \quad (\text{A9})$$

and

$$0 = - \mathbf{B} \cdot \nabla^0 \tilde{\Phi}^0. \quad (\text{A10})$$

From Eqs. (A9) and (A10) we conclude that

$$\tilde{\Phi}^0 = \tilde{\Phi}^0(\psi) = \langle \tilde{\Phi}^0 \rangle, \quad \tilde{p}^{-1} = \tilde{p}^{-1}(\psi) = \langle \tilde{p}^{-1} \rangle, \quad (\text{A11})$$

that is, $\tilde{\Phi}^0$ and \tilde{p}^{-1} are flux functions.

The equilibrium vorticity equation assumes the form

$$\mathbf{B} \cdot \nabla^0 \frac{J_{\parallel}}{B} + 2c \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \psi}{B} p_{\psi} = 0. \quad (\text{A12})$$

Thus, Eqs. (A6) and (A8) may be integrated to yield

$$\tilde{J}_{\parallel}^{-2} = (\sigma B - \langle \sigma B \rangle) \tilde{p}_{\psi}^{-1} / p_{\psi} + \langle \tilde{J}_{\parallel}^{-2} \rangle, \quad (\text{A13})$$

$$\tilde{v}_{\parallel}^{-1} = (\sigma B - \langle \sigma B \rangle) \tilde{\Phi}_{\psi}^0 / (4\pi p_{\psi}) + \langle \tilde{v}_{\parallel}^{-1} \rangle, \quad (\text{A14})$$

where $c\sigma B \equiv 4\pi J_{\parallel}$ and we have expressed the constant of integration in Eqs. (13) and (14) in terms of the averaged functions $\langle \tilde{J}_{\parallel}^{-2} \rangle$ and $\langle \tilde{v}_{\parallel}^{-1} \rangle$. Equations (A13) and (A14) yield the θ variation of $\tilde{J}_{\parallel}^{-2}$ and $\tilde{v}_{\parallel}^{-1}$ in terms of their field-line-averaged counterparts. The θ variation of \tilde{A}^0 can be calculated from Eqs. (A7) and (A13). To obtain equations for the averaged functions $\langle \tilde{\Phi}^0 \rangle, \dots$, we average the next-order versions of Eqs. (A1)–(A5) over θ . For example, the order $\tilde{\epsilon}^0$ and $\tilde{\epsilon}^1$ versions of Eqs. (A4) and (A5) become

$$\gamma \rho B \tilde{v}_{\parallel}^{-1} = -\mathbf{B} \cdot \nabla^0 \tilde{p}^0 - \mathbf{B} \cdot \nabla^1 \tilde{p}^{-1} - in B_0 p_{\psi} \tilde{A}^0, \quad (\text{A15})$$

$$DB \tilde{J}_{\parallel}^{-2} = -\mathbf{B} \cdot \nabla^0 \tilde{\Phi}^1 - \mathbf{B} \cdot \nabla^1 \tilde{\Phi}^0 + \gamma B_0 \tilde{A}^0. \quad (\text{A16})$$

The unknown functions $\tilde{\Phi}^1$ and \tilde{p}^0 are eliminated from these equations by integrating over θ and using the periodicity constraint. The resulting equations are

$$\gamma \rho \langle B \tilde{v}_{\parallel}^{-1} \rangle = -\langle \mathbf{B} \cdot \nabla^1 \rangle \langle \tilde{p}^{-1} \rangle - in B_0 p_{\psi} \langle \tilde{A}^0 \rangle, \quad (\text{A17})$$

$$D \langle B \tilde{J}_{\parallel}^{-2} \rangle = -\langle \mathbf{B} \cdot \nabla^1 \rangle \langle \tilde{\Phi}^0 \rangle + \gamma B_0 \langle \tilde{A}^0 \rangle. \quad (\text{A18})$$

We then combine Eqs. (A17) and (A18) with Eqs. (A13) and (A14) to obtain expressions for $\langle \tilde{v}_{\parallel}^{-1} \rangle$ and $\langle \tilde{J}_{\parallel}^{-2} \rangle$ in terms of $\langle \tilde{\Phi}^0 \rangle, \langle \tilde{p}^{-1} \rangle$, and $\langle \tilde{A}^0 \rangle$. We carry out the same procedure on Eqs. (A1) and (A3) to obtain equations for $\langle \tilde{\Phi}^0 \rangle$ and $\langle \tilde{p}^{-1} \rangle$:

$$\begin{aligned} \frac{\gamma 4\pi \rho}{B_0^2} \langle |\nabla \psi|^2 \rangle \langle \tilde{\Phi}^0 \rangle_{\psi\psi} \\ = \langle \mathbf{B} \cdot \nabla^1 \frac{\tilde{J}_{\parallel}^{-2}}{B} \rangle + in 8\pi \left\langle \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla u}{B} \right\rangle \langle \tilde{p}^{-1} \rangle \\ + 8\pi \left\langle \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \psi}{B} \tilde{p}_{\psi}^0 \right\rangle \end{aligned} \quad (\text{A19})$$

and

$$\begin{aligned} \gamma \langle \tilde{p}^{-1} \rangle + in p_{\psi} \langle \tilde{\Phi}^0 \rangle + \gamma_s p \left(\left\langle \mathbf{B} \cdot \nabla^1 \frac{\tilde{v}_{\parallel}^{-1}}{B} \right\rangle \right. \\ \left. + 2 \left\langle \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \psi}{B} \tilde{\Phi}_{\psi}^1 \right\rangle + 2in \left\langle \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla u}{B} \right\rangle \langle \tilde{\Phi}^0 \rangle \right. \\ \left. - \frac{4\pi}{B_0^2} D \langle |\nabla \psi|^2 \rangle \langle \tilde{p}^{-1} \rangle_{\psi\psi} \right) = 0. \end{aligned} \quad (\text{A20})$$

The terms involving $\tilde{\Phi}_{\psi}^1$ and \tilde{p}_{ψ}^0 in Eqs. (A19) and (A20) can be simplified using Eq. (A12). For example,

$$8\pi \left\langle \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla \psi}{B} \tilde{p}_{\psi}^0 \right\rangle = \frac{-\langle p_{\psi}^0 \mathbf{B} \cdot \nabla^0 \sigma \rangle}{p_{\psi}} = \frac{\langle \sigma \mathbf{B} \cdot \nabla^0 \tilde{p}^0 \rangle_{\psi}}{p_{\psi}}, \quad (\text{A21})$$

where we have commuted the ψ derivative in Eq. (A21) with the equilibrium quantities due to the fact that the dominant ψ dependence arises from the radial structure of perturbations. The final average in Eq. (A21) can be completed by multiplying Eq. (A15) by σ and averaging. Similar manipulations can be carried out to simplify Eq. (A20). The final equation

for $\langle \tilde{A}^0 \rangle$ is obtained by directly averaging Eq. (A7).

The detailed algebraic steps required to complete the averaging of Eqs. (A7) and (A17)–(A20) are lengthy but not difficult and are not presented here. The final result is a system of coupled differential equations for $\langle \tilde{\Phi}^0 \rangle, \langle \tilde{p}^{-1} \rangle$, and $\langle \tilde{A}^0 \rangle$,

$$\begin{aligned} \gamma (4\pi \rho / B_0^2) \langle |\nabla \psi|^2 \rangle (1 + T_1) \langle \tilde{\Phi} \rangle_{\psi\psi} \\ = D^{-1} ik_{\parallel} \langle \gamma \tilde{A} \rangle - ik_{\parallel} \langle \tilde{\Phi} \rangle \\ + in(4\pi G \langle \tilde{p} \rangle - T_2 R_0 p_{\psi} \langle \tilde{A} \rangle_{\psi}), \end{aligned} \quad (\text{A22})$$

$$D \langle |\nabla \psi|^{-2} \rangle^{-1} \langle \tilde{A} \rangle_{\psi\psi} = \gamma \langle \tilde{A} \rangle - ik_{\parallel} \langle \tilde{\Phi} \rangle + DT_2 R_0 \langle \tilde{p} \rangle_{\psi}, \quad (\text{A23})$$

and

$$\begin{aligned} \gamma \langle \tilde{p} \rangle - \frac{4\pi \gamma_s p}{B_0^2} D \left(1 + \frac{T_1}{2} \right) \langle |\nabla \psi|^2 \rangle \langle \tilde{p} \rangle_{\psi\psi} \\ + \frac{c_s^2}{\gamma} k_{\parallel} \langle k_{\parallel} \langle \tilde{p} \rangle + n p_{\psi} \langle \tilde{A} \rangle \rangle \\ + in(p_{\psi} - \gamma_s p G) \langle \tilde{\Phi} \rangle + \gamma \gamma_s p T_2 R_0 \langle \tilde{A} \rangle_{\psi} = 0, \end{aligned} \quad (\text{A24})$$

where $c_s = (\gamma_s p / \rho)^{1/2}$ is the sound speed,

$$k_{\parallel} = \frac{1}{i B_0} \langle \mathbf{B} \cdot \nabla^1 \rangle = \frac{n}{R_0 q} \left(q - \frac{m}{n} \right), \quad (\text{A25})$$

$$G = 2 \left\langle \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla u}{B} \right\rangle - 4\pi p_{\psi} T_4 \langle B_p^{-2} \rangle + \frac{T_3 q_{\psi}}{q}, \quad (\text{A26})$$

and the superscripts on the perturbed quantities have been dropped for simplicity. The quantities T_1 – T_4 are dimensionless coefficients which are nonzero in toroidal geometry,

$$T_1 = \frac{(4\pi)^{-2} B_0^4}{\langle |\nabla \psi|^2 \rangle P_{\psi}^2} \left(\frac{\langle B^2 \sigma^2 \rangle}{\langle B^2 \rangle} - \frac{\langle \sigma B^2 \rangle^2}{\langle B^2 \rangle^2} \right) \sim q^2, \quad (\text{A27})$$

$$T_2 = -\frac{(4\pi)^{-1} B_0}{R_0 P_{\psi}} \left(\frac{\langle B^2 \sigma \rangle}{\langle B^2 \rangle} - \frac{\langle \langle B^2 \sigma \rangle \rangle}{\langle \langle B^2 \rangle \rangle} \right) \sim \epsilon^2, \quad (\text{A28})$$

$$T_3 = \frac{(4\pi)^{-1} B_0}{R_0 P_{\psi}} \left(\frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} - \langle \sigma \rangle \right) \sim \epsilon^2, \quad (\text{A29})$$

and

$$T_4 = \frac{(4\pi)^{-2}}{R_0^2 P_{\psi}^2} \left(\langle \langle B^2 \sigma^2 \rangle \rangle - \frac{\langle \langle B^2 \sigma \rangle \rangle^2}{\langle \langle B^2 \rangle \rangle} \right) \sim \epsilon^2, \quad (\text{A30})$$

where

$$\langle \langle A \rangle \rangle \equiv \langle |\nabla \psi|^{-2} A \rangle / \langle |\nabla \psi|^{-2} \rangle.$$

The coefficient T_1 in the vorticity equation and the cross-field particle diffusion coefficient is the Pfirsch–Schlüter enhancement of the inertia in toroidal geometry which results from the pressure-driven parallel flow. These equations are consistent with the low- β , large-aspect-ratio limit of the equations presented in Ref. 9.

Finally, to obtain the Mercier criterion, we take the limit $\gamma, D \rightarrow 0$. From (A23) and (A24), we obtain

$$\gamma \langle \tilde{A} \rangle - ik_{\parallel} \langle \tilde{\Phi} \rangle = 0,$$

$$n p_{\psi} \langle \tilde{A} \rangle + k_{\parallel} \langle \tilde{p} \rangle = 0,$$

and we then expand k_{\parallel} around the rational surface ψ_r , $k_{\parallel} = k_{\parallel\psi}(\psi - \psi_r)$, to obtain the equation

$$\langle \tilde{A} \rangle_{\psi\psi} + [D_M / (\psi - \psi_r)^2] \langle \tilde{A} \rangle = 0, \quad (\text{A31})$$

where

$$D_M = \frac{q^2}{q_\psi^2} \langle B_p^{-2} \rangle \left[\frac{q_\psi}{q} \frac{B_0}{R_0} \left(\langle \sigma \rangle - \frac{\langle \langle B^2 \sigma \rangle \rangle}{\langle \langle B^2 \rangle \rangle} \right) + B_0^2 \langle |\nabla \psi|^{-2} \rangle \left(\frac{\langle \langle B^2 \sigma^2 \rangle \rangle}{\langle \langle B^2 \rangle \rangle} - \frac{\langle \langle B^2 \sigma \rangle \rangle^2}{\langle \langle B^2 \rangle \rangle^2} \right) - 2p_\psi \left\langle \frac{\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla u}{B} \right\rangle \right]. \quad (\text{A32})$$

Stability requires that the solutions to (A31) be nonoscillatory or

$$D_M < \frac{1}{4}. \quad (\text{A33})$$

Equations (A32) and (A33) constitute the Mercier criterion for a low- β , large-aspect-ratio tokamak, where $B = B_0 R_0 / R$, \mathbf{B} and $(\mathbf{b} \times \boldsymbol{\kappa})/B$ are given in (20) and (25) and ψ and $4\pi J_{\parallel} / Bc$ are solutions of the equilibrium equations

$$R^2 \nabla \cdot \mathbf{R}^{-2} \nabla \psi = R_0 B_0 \sigma, \quad (\text{A34})$$

$$\mathbf{B} \cdot \nabla \sigma + 8\pi p_\psi [(\mathbf{b} \times \boldsymbol{\kappa})/B] \cdot \nabla \psi = 0. \quad (\text{A35})$$

To lowest order the pressure term in (A35) can be neglected and $\sigma = \sigma_0(\psi)$. Corrections to σ can then be calculated by including the pressure,

$$\sigma = \sigma_0(\psi) - 8\pi p_\psi (R - R_0) / B_0 R_0. \quad (\text{A36})$$

The equilibrium flux then satisfies the equation

$$R^2 \nabla \cdot \mathbf{R}^{-2} \nabla \psi = B_0 R_0 \sigma_0(\psi) - 8\pi p_\psi (R - R_0). \quad (\text{A37})$$

Both Eq. (A37) for ψ and Eq. (A36) for σ are valid through first order in ϵ as are our previous expressions for B and $\mathbf{b} \times \boldsymbol{\kappa} / B$. We will now show that these equilibrium quantities are sufficiently accurate to evaluate the Mercier criterion in (A33).

Each of the three terms on the right side of (A32) are of the same order. In the expression for the average curvature, the lowest-order toroidal curvature averages to zero ($\langle \hat{z} \cdot \nabla \theta \rangle \approx 0$). The lowest-order surviving terms (such as the poloidal curvature) are of order ϵ smaller so that

$$p_\psi \langle (\mathbf{b} \times \boldsymbol{\kappa} \cdot \nabla u) / B \rangle \sim R_0^{-4}.$$

Since we have included corrections of order ϵ to all of our equilibrium quantities, the average curvature can be evaluated to the requisite order. That the same is true for the two other terms in (A32) is perhaps less obvious. In the shear term in (A32),

$$\langle \sigma \rangle - \frac{\langle \langle B^2 \sigma \rangle \rangle}{\langle \langle B^2 \rangle \rangle} \sim \sigma \epsilon^2, \quad (\text{A38})$$

thus it superficially appears as if corrections to σ and B^2 of order ϵ^2 are required to evaluate this expression. This is not the case. Each of the terms in Eq. (A38) can be expanded as a power series in the aspect ratio as

$$\begin{aligned} \sigma &= \sigma_0 + \epsilon \sigma_1 \cos \chi + \epsilon^2 \sigma_2 \cos 2\chi + \epsilon^2 \sigma_2^2 + \dots, \\ B &= B_0 + \epsilon B_1 \cos \chi + \epsilon^2 B_2 \cos 2\chi + \epsilon^2 B_2^2 + \dots \end{aligned}$$

Such an expansion was carried out, for example, in Ref. 10. The field line average $\int dl / B$ can similarly be expressed as an integral over the poloidal angle χ . The only terms which survive the averaging process involve cross products of the χ dependence of two quantities such as $\sigma_1 B_1$. Thus, to evaluate (A38) and the remainder of (A32) only corrections to σ , B , and ψ of order ϵ must be retained.

For a large-aspect-ratio torus with shifted circular flux

surfaces, D_M can be evaluated explicitly as in Ref. 10:

$$D_M = \frac{2r}{R_0^2} L_s^2 \frac{1}{B_0^2} \frac{dp}{dr} \left(1 - \frac{1}{q^2(r)} \right),$$

where $L_s = q^2 R_0 / (r dq/dr)$ is the magnetic shear length. Since for normal profiles $dp/dr < 0$, the average curvature is favorable for $q > 1$. Moreover within our ordering

$$D_M \sim \epsilon^2,$$

For $dq/dr \sim r^{-1}$ so that the Mercier criterion is usually satisfied even for $q < 1$. In a region where $q < 1$ and $dq/dr \rightarrow 0$, instability is possible. In any case our reduced nonlinear equations correctly include the average favorable magnetic curvature in a torus with $q > 1$.

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